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Nonlinear particle kinematics of ocean waves†

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A fundamental relation is derived governing the Lagrangian kinematics of fluid particles on the surface of nonlinear ocean waves which may be known only stochastically. The horizontal trajectories of fluid particles on the free surface are shown to obey a pair of coupled nonlinear Ricatti-type ordinary differential equations driven by the temporal and spatial gradients of the free-surface elevation defined relative to an Eulerian frame. This equation is explicit in that it does not require the solution of a fully nonlinear potential flow free-surface problem and may be viewed as a deterministic or stochastic equation depending on the interpretation of the definition of the free-surface elevation. It is free of empirical corrections often used to estimate the particle kinematics above the calm water surface, is valid in potential flow and for waves of large steepness in two and three dimensions and in waters of all depths and may be used for the evaluation of the extreme unsteady loads exerted on surface piercing vertical circular cylinders by steep random waves.

1. Introduction

The extreme wave loads exerted upon fixed offshore platforms by severe sea states are known to depend in a critical manner on the kinematics of fluid particles in the surf zone. Such loads are known to be responsible for the ringing responses of bottom mounted, tension leg and other offshore platforms, yet their satisfactory theoretical modelling is yet unavailable. Empirical corrections of the wave particle velocities and accelerations above the calm water surface are often used (see Wheeler 1970), yet their accuracy is limited in highly nonlinear, unsteady and random wave conditions. This paper resolves these issues by deriving an explicit model for the nonlinear kinematics of fluid particles on the exact position of a three-dimensional ocean surface conditional upon its evolution in a deterministic or stochastic manner by circumventing the need to solve a fully nonlinear potential flow free-surface problem.

John (1953) (see also Wehausen & Laitone 1960, p. 740) derives an exact relation between the fully nonlinear kinematics of fluid particles on the surface of two-dimensional waves to the prescribed evolution of their elevation. John derives this result in two dimensions by employing complex variable analytic methods. The present note re-derives this result without the use of complex variables and extends it to three dimensions. The horizontal trajectories of free-surface fluid particles are shown to obey a pair of coupled nonlinear Ricatti-type ordinary differential equations driven by the Eulerian temporal and spatial gradients of the free-surface elevation. This

† With an Appendix by T. J. Bridges.

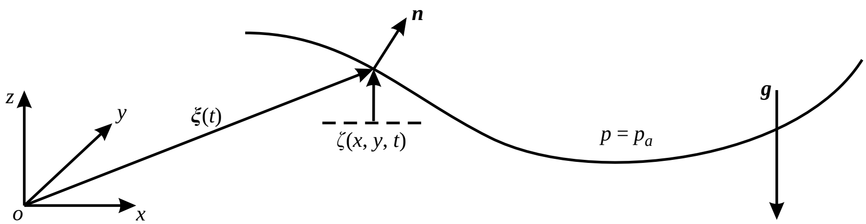

FIGURE 1. Eulerian and Lagrangian coordinates of a fluid particle on the free surface.

equation is explicit in that it does not require the solution of a fully nonlinear potential flow free-surface problem.

In an Appendix contributed by Thomas J. Bridges, this Ricatti equation is shown to accept singular solutions. The interpretation of such a singular behaviour in seastates which are known only stochastically is discussed and Monte Carlo simulations are suggested as a means of evaluating summary statistics and probability density functions of the nonlinear particle kinematics.

The application of these results to the fully nonlinear loading of slender vertical cylinders in steep random waves may prove valuable. The particle velocities and accelerations on the free surface obtained from the solution of the Ricatti equation include both unsteady and convective components with no empirical corrections of the wave particle kinematics above the calm water surface being necessary. A numerical and experimental study by Swan, Bashir & Gudmestad (2002) confirms that linear random wave theory equipped with empirical corrections for the particle kinematics above the calm water plane is not capable of modelling accurately the highly nonlinear wave loads on surface piercing structures. The Swan *et al.* study further confirms that the accuracy of nonlinear wave load predictions depends in a critical manner on the accurate modelling of the unsteady water particle accelerations. Therefore, the theory presented in the present paper may prove useful in the estimation of extreme wave loads on offshore structures when coupled with potential wave loading models for slender vertical cylinders as derived by Rainey (1995) and Faltinsen, Newman & Vinje (1994). Moreover, the model presented in the present article for the wave particle kinematics offers a framework for the study of wave breaking in a stochastic wave environment.

2. Free-surface particle evolution equations

Figure 1 plots a three-dimensional free surface assumed single valued and defined by

$$z = \zeta(x, y, t). \tag{2.1}$$

The instantaneous vector position of a fluid particle on the free surface is denoted by $\boldsymbol{\xi}(t)$, the acceleration due to gravity is a downwards pointing vector $\boldsymbol{g} = (0, 0, -g)$, the atmospheric pressure is a constant equal to p_a and $\boldsymbol{n}$ is a unit normal vector normal to the free surface and pointing out of the fluid domain.

Euler's equation in the fluid domain and on the free surface, states

$$\frac{\mathrm{d}^2\boldsymbol{\xi}}{\mathrm{d}t^2} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}. \tag{2.2}$$

The function

$$F(x, y, z, t) = z - \zeta(x, y, t) \tag{2.3}$$

vanishes on the free surface by definition, therefore its gradient

$$\nabla F = \left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right) \tag{2.4}$$

is a vector normal to the free surface, hence colinear with the unit normal vector $\boldsymbol{n}$.

The hydrodynamic pressure p is also constant on the free surface and equal to its atmospheric value. Therefore its gradient ∇p is also normal to the free surface and colinear with ∇F. It follows that the exterior product of ∇F and ∇p vanishes when evaluated on the free surface, or

$$\nabla F \times \nabla p = 0 \qquad \text{on} \quad z = \zeta(x, y, t). \tag{2.5}$$

By virtue of Euler's equation (2.2) and the definition given by (2.4), it follows that on $z = \zeta(x, y, t)$ the following vector identity holds:

$$\left(-\frac{\partial \zeta}{\partial x}, -\frac{\partial \zeta}{\partial y}, 1\right) \times \left(\frac{\mathrm{d}^2 \boldsymbol{\xi}}{\mathrm{d}t^2} - \boldsymbol{g}\right) = 0. \tag{2.6}$$

Upon expansion in its x-, y- and z-components, three identities follow:

$$\frac{\partial \zeta}{\partial y}\left(\frac{\mathrm{d}^2 \xi_3}{\mathrm{d}t^2} + g\right) + \frac{\mathrm{d}^2 \xi_2}{\mathrm{d}t^2} = 0, \tag{2.7}$$

$$\frac{\partial \zeta}{\partial x}\left(\frac{\mathrm{d}^2 \xi_3}{\mathrm{d}t^2} + g\right) + \frac{\mathrm{d}^2 \xi_1}{\mathrm{d}t^2} = 0, \tag{2.8}$$

$$\frac{\partial \zeta}{\partial x}\frac{\mathrm{d}^2 \xi_2}{\mathrm{d}t^2} - \frac{\partial \zeta}{\partial y}\frac{\mathrm{d}^2 \xi_1}{\mathrm{d}t^2} = 0. \tag{2.9}$$

The set of equations (2.7)–(2.8) represents a coupled system of equations relating the evolution of the particle trajectories to the spatial gradients of the free-surface elevation $\zeta(x, y, t)$ in three dimensions. Equation (2.9) may be seen to be redundant since it follows as the difference of (2.7) multiplied by $\partial \zeta/\partial x$ and (2.8) multiplied by $\partial \zeta/\partial y$.

Setting $\xi_2(t) = 0$, $\partial \zeta/\partial y = 0$, restricts the problem to the propagation of surface waves in two dimensions and in the positive or negative x-direction. In this special but important case, only (2.8) is non-trivial. It must be complemented by the consistency condition

$$\xi_3(t) = \zeta\,[x = \xi_1(t), t] = \zeta\,[\xi_1(t), t]. \tag{2.10}$$

Derivatives of $\xi_3(t)$ with respect to time may now be taken formally using (2.10) and then substituted into (2.8). A straightforward application of the chain rule yields:

$$\frac{\mathrm{d}\xi_3}{\mathrm{d}t} = \frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}, \tag{2.11}$$

$$\frac{\mathrm{d}^2 \xi_3}{\mathrm{d}t^2} = \frac{\partial^2 \zeta}{\partial t^2} + 2\frac{\partial^2 \zeta}{\partial x \partial t}\frac{\mathrm{d}\xi_1}{\mathrm{d}t} + \frac{\partial^2 \zeta}{\partial x^2}\left(\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\right)^2 + \frac{\partial \zeta}{\partial x}\frac{\mathrm{d}^2 \xi_1}{\mathrm{d}t^2}. \tag{2.12}$$

Upon substitution of (2.12) into (2.8), a nonlinear ordinary equation relating the horizontal particle displacement $\xi_1(t)$ in terms of spatial and temporal gradients of the free-surface elevation $\zeta(x,t)$ follows:

$$\left[1+\left(\frac{\partial\zeta}{\partial x}\right)^2\right]\frac{\mathrm{d}^2\xi_1}{\mathrm{d}t^2}+2\frac{\partial\zeta}{\partial x}\frac{\partial^2\zeta}{\partial x\partial t}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}+\frac{\partial\zeta}{\partial x}\frac{\partial^2\zeta}{\partial x^2}\left(\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\right)^2=-\frac{\partial\zeta}{\partial x}\left(\frac{\partial^2\zeta}{\partial t^2}+g\right). \quad (2.13)$$

Equation (2.13) is the result derived by John (1953) using potential flow analytic methods.

The two-dimensional result derived above may be extended to three dimensions by first invoking the more general consistency condition

$$\xi_3(t)=\zeta(\xi_1(t),\xi_2(t),t). \quad (2.14)$$

It follows that

$$\frac{\mathrm{d}\xi_3}{\mathrm{d}t}=\frac{\partial\zeta}{\partial t}+\frac{\partial\zeta}{\partial x}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}+\frac{\partial\zeta}{\partial y}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}. \quad (2.15)$$

After some algebra

$$\begin{aligned}\frac{\mathrm{d}^2\xi_3}{\mathrm{d}t^2}=&\frac{\partial^2\zeta}{\partial t^2}+2\frac{\partial^2\zeta}{\partial x\partial t}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}+2\frac{\partial^2\zeta}{\partial y\partial t}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}+\frac{\partial^2\zeta}{\partial x^2}\left(\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\right)^2+\frac{\partial^2\zeta}{\partial y^2}\left(\frac{\mathrm{d}\xi_2}{\mathrm{d}t}\right)^2\\&+\frac{\partial\zeta}{\partial x}\frac{\mathrm{d}^2\xi_1}{\mathrm{d}t^2}+\frac{\partial\zeta}{\partial y}\frac{\mathrm{d}^2\xi_2}{\mathrm{d}t^2}+2\frac{\partial^2\zeta}{\partial x\partial y}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}.\end{aligned} \quad (2.16)$$

Upon substitution into (2.8) the following three-dimensional extensions to (2.13) follow

$$\begin{aligned}&\left[1+\left(\frac{\partial\zeta}{\partial x}\right)^2\right]\frac{\mathrm{d}^2\xi_1}{\mathrm{d}t^2}+\frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y}\frac{\mathrm{d}^2\xi_2}{\mathrm{d}t^2}+2\frac{\partial\zeta}{\partial x}\frac{\partial^2\zeta}{\partial x\partial y}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}\\&\quad+\frac{\partial\zeta}{\partial x}\left[2\frac{\partial^2\zeta}{\partial x\partial t}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}+2\frac{\partial^2\zeta}{\partial y\partial t}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}+\frac{\partial^2\zeta}{\partial x^2}\left(\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\right)^2+\frac{\partial^2\zeta}{\partial y^2}\left(\frac{\mathrm{d}\xi_2}{\mathrm{d}t}\right)^2\right]\\&\quad=-\frac{\partial\zeta}{\partial x}\left(\frac{\partial^2\zeta}{\partial t^2}+g\right),\end{aligned} \quad (2.17)$$

$$\begin{aligned}&\left[1+\left(\frac{\partial\zeta}{\partial y}\right)^2\right]\frac{\mathrm{d}^2\xi_2}{\mathrm{d}t^2}+\frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y}\frac{\mathrm{d}^2\xi_1}{\mathrm{d}t^2}+2\frac{\partial\zeta}{\partial y}\frac{\partial^2\zeta}{\partial x\partial y}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\frac{\mathrm{d}\xi_2}{\mathrm{d}t}\\&\quad+\frac{\partial\zeta}{\partial y}\left[2\frac{\partial^2\zeta}{\partial x\partial t}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}+2\frac{\partial^2\zeta}{\partial x\partial t}\frac{\mathrm{d}\xi_1}{\mathrm{d}t}+\frac{\partial^2\zeta}{\partial x^2}\left(\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\right)^2+\frac{\partial^2\zeta}{\partial y^2}\left(\frac{\mathrm{d}\xi_2}{\mathrm{d}t}\right)^2\right]\\&\quad=-\frac{\partial\zeta}{\partial y}\left(\frac{\partial^2\zeta}{\partial t^2}+g\right).\end{aligned} \quad (2.18)$$

The nonlinear system of equations (2.17) and (2.18) extends John's result to three dimensions.

3. Solution of particle evolution equations

Equations (2.13) in two dimensions and (2.17)–(2.18) in three accept as input the ambient wave elevation surface $\zeta(x,y,t)$ specified in a deterministic or stochastic manner. This is the approach followed in most analysis and design studies involving the interaction of ocean waves and floating structures.

For the sake of clarity, the two-dimensional case will be discussed in the present section. Extensions to the three-dimensional case, which is more important in practice, are conceptually obvious, but computationally more tedious.

The two-dimensional evolution equation (2.13) may be recast in the form

$$\frac{\mathrm{d}^2\xi_1}{\mathrm{d}t^2} + A(\xi_1, t)\frac{\mathrm{d}\xi_1}{\mathrm{d}t} + B(\xi_1, t)\left(\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\right)^2 = C(\xi_1, t), \tag{3.1}$$

where

$$A(\xi_1, t) = \left.\frac{2\dfrac{\partial \zeta}{\partial x}\dfrac{\partial^2 \zeta}{\partial x \partial t}}{1 + \left(\dfrac{\partial \zeta}{\partial x}\right)^2}\right|_{x=\xi_1}, \tag{3.2}$$

$$B(\xi_1, t) = \left.\frac{\dfrac{\partial \zeta}{\partial x}\dfrac{\partial^2 \zeta}{\partial x^2}}{1 + \left(\dfrac{\partial \zeta}{\partial x}\right)^2}\right|_{x=\xi_1}, \tag{3.3}$$

$$C(\xi_1, t) = \left.\frac{-\dfrac{\partial \zeta}{\partial x}\left(\dfrac{\partial^2 \zeta}{\partial t^2} + g\right)}{1 + \left(\dfrac{\partial \zeta}{\partial x}\right)^2}\right|_{x=\xi_1}. \tag{3.4}$$

Equation (3.1) is a Ricatti-type equation with time-dependent coefficients defined by (3.2)–(3.4) which in turn are functions of the prescribed evolution of the ambient wave surface $\zeta(x, t)$ defined as a function of its Eulerian coordinates (x, t).

An in-depth analysis of the solution properties of the Ricatti equation has been graciously contributed by Thomas J. Bridges and is detailed in the Appendix. The analysis in the Appendix demonstrates that the solution of (3.1)–(3.4) for the horizontal particle trajectories may have a potentially very singular behaviour. Moreover, it is shown in the Appendix that the solution is very sensitive to the selection of the initial conditions.

As is the case with the solution of other nonlinear equations, the selection of the appropriate initial conditions may be far from a trivial matter. In the present problem, the existence of singular solutions may be an indication of the presence of a physical instability in the particle evolution equations which may offer insights into the incipience of wave breaking. The analytical proof of the existence of singular solutions offered in the Appendix confirms that their origin is not of a numerical nature as would be the case with certain computational algorithms for the simulation of evolution of nonlinear surface wave trains. Moreover, the transformation of the Ricatti equation (3.1) into a linear Hamiltonian ODE with time-dependent coefficients in the Appendix (equation (A8)), reveals analogies with Mathieu type equations which offer valuable insights into other complex stability problems, for example the capsizing of vessels in steep random waves, in spite of their approximate nature.

As pointed out in the Appendix, the solution of the free-surface particle path trajectories will be well behaved, if the initial horizontal particle velocity is correctly specified which may require the solution of the fully nonlinear free-surface problem.

The insights gained from the derivation of (3.1)–(3.4) and the analysis in the Appendix, however, suggest an alternative stochastic interpretation and use of the particle evolution Ricatti equation. In a realistic seastate, the free-surface elevation is, however, known only stochastically. One possible representation of a stochastic seastate is based on the use of perturbation theory up to third order, as developed in Sclavounos (1992).

An alternative stochastic characterization of the ambient wave elevation $\zeta(x,t)$ in narrow-banded sea states is given by the expression

$$\zeta(x,t) = \mathrm{Re}\{\rho(t)\exp(\mathrm{i}\,[\omega(t)t - k(\omega)x])\}, \tag{3.5}$$

where in a linear setting the relation between the wavenumber and wave frequency in deep water is given by the dispersion relation $k=\omega^2/g$. Randomness in (3.5) is introduced by assuming that:

$$\omega(t) = \omega_0 t + \varphi(t), \tag{3.6}$$

where ω_0 is a deterministic carrier frequency and $\varphi(t)$ is a slowly varying random phase, namely, $\mathrm{d}\varphi/\mathrm{d}t \ll \omega_0$ for all t. The modulus $\rho(t)$ is also random, positive and slowly varying. The joint probability density function of the random variables $(\rho, \varphi, \mathrm{d}\rho/\mathrm{d}t, \mathrm{d}\varphi/\mathrm{d}t)$ corresponding to a given wave spectral density has been derived by Longuet-Higgins (1983) and found to be in very good agreement with experimental measurements. The temporal and spatial derivatives of the ambient wave elevation that appear in the coefficients of the Ricatti equation (3.1)–(3.4) may be readily obtained from the definitions (3.5) and (3.6).

In this stochastic setting, initial conditions to the Ricatti equation are random variables drawn from the joint distribution discussed above. Given a sample drawn from this distribution, the solution of the Ricatti equation may proceed in a deterministic manner, and as pointed out in the Appendix may develop a singular behaviour for particular samples. However, summary statistics of the horizontal particle kinematics, for example, the mean or variance of the horizontal particle velocity or acceleration may be finite since these are merely integrals of their values weighed by their probability density functions. Such statistics may be estimated by Monte Carlo simulations each involving the solution of the Ricatti equation (3.1)–(3.4) for each sample. Singular solutions of the Ricatti equation may exist for particular samples, yet they may turn out to be integrable when summary statistics are evaluated.

Simulations along the lines suggested above may also be used for the derivation of the probability density function of the horizontal particle kinematics in stochastic seastates. The form of these probability distributions may contain useful insights on the nature of wave breaking and the likelihood of its occurrence in random waves, particularly when a subset of the samples lead to singular solutions of the Ricatti equation.

Knowledge of the statistics of the horizontal particle kinematics discussed above may also be useful in the evaluation of the inertial loading of vertical cylinders in steep random waves. These loads are known to depend in a critical manner on the horizontal acceleration of fluid particles in the surfzone. The values predicted by the present theory on the actual position of the free surface are bound to be the highest, therefore errors arising from the interpolation into the fluid domain are likely not to be material since the exact physics governing the particle kinematics is contained in (3.1)–(3.4) and its three dimensional extensions (2.17)–(2.18).

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Appendix. Potentially singular particle paths

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A.1. *The John–Sclavounos equations*

The purpose of this Appendix is to show that the equations derived by Sclavounos for the horizontal particle positions (2.17)–(2.18) can have potentially very singular behaviour.

First, write these equations in a convenient form which illustrates their structure and clarifies their analysis. Shift $\zeta(x, y, t) = \eta(x, y, t) + gt^2/2$ and let

$$u_1(t) = \frac{\mathrm{d}\xi_1}{\mathrm{d}t}, \quad u_2(t) = \frac{\mathrm{d}\xi_2}{\mathrm{d}t}, \quad \boldsymbol{B}(t) = \begin{bmatrix} 1+\eta_x^2 & \eta_x\eta_y \\ \eta_x\eta_y & 1+\eta_y^2 \end{bmatrix}.$$

Then (2.17)–(2.18) can be reformulated as

$$\boldsymbol{B}(t)\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + Q(\boldsymbol{u}, t)\begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} = 0, \quad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{A 1}$$

where $Q(\boldsymbol{u}, t)$ is the scalar-valued quadratic form

$$Q(\boldsymbol{u}, t) = \left\langle \begin{pmatrix} u_1 \\ u_2 \\ 1 \end{pmatrix}, \boldsymbol{C}(t)\begin{pmatrix} u_1 \\ u_2 \\ 1 \end{pmatrix} \right\rangle \quad \text{with} \quad \boldsymbol{C}(t) = \begin{bmatrix} \eta_{xx} & \eta_{xy} & \eta_{xt} \\ \eta_{yx} & \eta_{yy} & \eta_{yt} \\ \eta_{tx} & \eta_{ty} & \eta_{tt} \end{bmatrix}. \tag{A 2}$$

The matrix $\boldsymbol{C}(t)$ is a generalized curvature matrix. Generalized in that it views $\eta(x, y, t)$ as a graph over all three dimensions: x, y and t, and therefore includes curvature in time as well as space.

The matrix $\boldsymbol{B}(t)$ is always non-singular. Moreover, $(\eta_x, \eta_y)^T$ is an eigenvector of $\boldsymbol{B}(t)$ with eigenvalue equal to the determinant of $\boldsymbol{B}(t)$. Hence,

$$\boldsymbol{B}(t)\begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} = \det(\boldsymbol{B}(t))\begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} = \left(1 + \eta_x^2 + \eta_x^2\right)\begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix},$$

and so,

$$\boldsymbol{B}(t)^{-1}\begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix} = \frac{1}{1+\eta_x^2+\eta_y^2}\begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix}.$$

Therefore, (2.17)–(2.18) can be written in the form

$$\boldsymbol{u}_t + \frac{Q(\boldsymbol{u}, t)}{1+\eta_x^2+\eta_y^2}\nabla\eta = 0 \quad \text{where} \quad \nabla\eta = \begin{pmatrix} \eta_x \\ \eta_y \end{pmatrix}. \tag{A 3}$$

There are several special cases of this system. Clearly, if $\eta_y = 0$, then $u_2(t)$ is a constant, and the system decouples into a single equation for $u_1(t)$. Similarly if $\eta_x = 0$, it reduces to a single equation for $u_2(t)$.

Other simplifications of (A 3) are possible when conditions are put on the coefficients. For example, if $\zeta_x^2 + \zeta_y^2 \neq 0$, then the transformation

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{bmatrix} \eta_x & -\eta_y \\ \eta_y & \eta_x \end{bmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$

reduces (A 3) to a coupled equation, one of which is linear and the other a Riccati equation. With other conditions on the coefficients, the coupled nonlinear equations can be reduced to coupled linear equations (see Enders & Schmidtmann 1992 for transformations of this type).

A.2. *Singular solutions of planar paths*

Consider the special case $\eta_y = 0$. Then $u_2(t)$ is constant, and without loss of generality can be taken to be zero. Then the coupled system (2.17)–(2.18) reduces to a single equation for $u(t) := u_1(t)$

$$u_t + \frac{2\zeta_x\zeta_{xt}}{1+\zeta_x^2}u + \frac{\zeta_x\zeta_{xx}}{1+\zeta_x^2}u^2 + \frac{\zeta_x(\zeta_{tt}+g)}{1+\zeta_x^2} = 0. \tag{A 4}$$

This is a Riccati equation and it can always be transformed to a linear ordinary differential equation (ignoring for the moment that ζ depends on ξ_1). However, the classical Riccati transformation (Reid 1972) fails because the coefficient of the nonlinear term, $\zeta_x\zeta_{xx}$, will in general change sign for any interesting wave fields. However, we can proceed more generally.

Replace ζ by η as defined above, and introduce the reduced curvature matrix,

$$\mathbf{C}(t) = \begin{bmatrix} \eta_{xx} & \eta_{xt} \\ \eta_{tx} & \eta_{tt} \end{bmatrix}, \quad q(u,t) = \begin{pmatrix} u \\ 1 \end{pmatrix}^T \mathbf{C}(t) \begin{pmatrix} u \\ 1 \end{pmatrix}. \tag{A 5}$$

Then (A 4) can be written,

$$u_t + \frac{q(u,t)}{1+\eta_x^2}\eta_x = 0. \tag{A 6}$$

Now introduce

$$u = v_1/v_2. \tag{A 7}$$

Then substitution into (A 4) leads to the following linear Hamiltonian ODE (again neglecting the ξ_1-dependence of ζ)

$$\boldsymbol{J}\boldsymbol{v}_t + \frac{\eta_x}{1+\eta_x^2}\mathbf{C}(t)\boldsymbol{v} = 0 \quad \text{where} \quad \boldsymbol{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{A 8}$$

This ODE is Hamiltonian since $\boldsymbol{J}$ is skew-symmetric and $\mathbf{C}(t)$ is symmetric. Note the central role played by the curvature matrix. Equation (A 8) is derived as follows. Substitute (A 7) into (A 6),

$$\frac{\dot{v}_1}{v_2} - \frac{v_1\dot{v}_2}{v_2^2} + \frac{1}{v_2^2}\frac{\eta_x}{1+\eta_x^2}\langle \boldsymbol{v}, \mathbf{C}(t)\boldsymbol{v}\rangle = 0, \tag{A 9}$$

or

$$\langle \boldsymbol{v}, \boldsymbol{J}\boldsymbol{v}_t\rangle + \frac{\eta_x}{1+\eta_x^2}\langle \boldsymbol{v}, \mathbf{C}(t)\boldsymbol{v}\rangle = 0, \tag{A 10}$$

where $\langle\cdot,\cdot\rangle$ is a standard inner product on $\mathbf{R}^2$. This equation is satisfied if

$$\boldsymbol{J}\boldsymbol{v}_t + \frac{\eta_x}{1+\eta_x^2}\mathbf{C}(t)\boldsymbol{v} \in \mathrm{Ker}(\boldsymbol{v}^T) = \gamma(t)\boldsymbol{J}\boldsymbol{v},$$

with $\gamma(t)$ arbitrary. By introducing an exponential transformation

$$\boldsymbol{v} \longmapsto \exp\left(\int_0^t \gamma(s)\,\mathrm{d}s\right)\boldsymbol{v},$$

the γ term can be transformed away. Hence the equation for $\boldsymbol{v}(t)$ is the ODE (A 8).

The behaviour of the solutions of (A 8) is determined principally by the determinant of $\boldsymbol{C}(t)$. Now, the determinant of $\boldsymbol{C}(t)$ is proportional to the Gaussian curvature of the surface $\zeta(x,t)$ over space *and time*. It is this generalized curvature which determines the behaviour of solutions.

To see the role of the determinant of $\boldsymbol{C}(t)$, consider the special case – for illustration – where $(\eta_x/1+\eta_x^2)\boldsymbol{C}(t)$ is a constant matrix, say $\boldsymbol{D}$,

$$\boldsymbol{J}\boldsymbol{v}_t + \boldsymbol{D}\boldsymbol{v} = 0.$$

The solutions of this equation are determined by the eigenvalues of $\boldsymbol{JD}$ which are $\pm\sqrt{-\det(\boldsymbol{D})}$. When the determinant is negative, the solutions are pure exponentials. However, when the determinant of $\boldsymbol{D}$ is positive, the components of $\boldsymbol{v}$ are sinusoidal and will therefore have a countable number of zeros. This latter observation leads to singular behaviour since $u = v_1/v_2$ and therefore when v_2 passes through zero, the velocity u has a singularity.

Even though the non-constant coefficient case is more complicated and cannot be analysed explicitly, it is clear that the solutions of (A 8) will have zeros in general, and may very well have many zeros. To summarize, the special case of (2.17)–(2.18) given in (A 4) can be expected to have solutions which blow up in finite time for a wide range of given free-surface positions $\zeta(x, y, t)$.

There is an inherent contradiction in this result. Given a smooth surface, defined by $z = \zeta(x, y, t)$, how is it that particle paths on the free surface can become singular? This contradiction will be addressed in the next section.

Equation (A 8) is in fact nonlinear. The nonlinearity arises because ζ (and therefore η) depends on $\xi_1(t)$ which in turn is related to u through $u = \dot{\xi}_1$. However, this nonlinearity will not, in general, remove the singularity – indeed it is more likely that the nonlinearity will enhance the singular behaviour.

A.3. *The role of initial conditions*

Riccati equations are very sensitive to initial conditions. For example, consider the simplest Riccati equation

$$u_t = u^2, \quad u(0) = u_0,$$

which has the exact solution

$$u(t) = \frac{u_0}{1 - u_0\, t}.$$

For positive initial data, all solutions blow up in finite time, with the time of blowup inversely proportional to the norm of the initial data. On the other hand, all solutions with negative initial data exist for all time. If the coefficient of the nonlinear term is non-constant, for example $u_t = a(t)u^2$, the set of initial data for which solutions exist for all time may be even more restricted.

It is clear that we must be careful in choosing initial data for (2.17)–(2.18) or (A 4). Consider the natural initial data for (A 4) (with obvious generalization to (2.17)–(2.18)). The Lagrangian particle paths, $\xi_1(t,a)$ and $\xi_2(t,a)$, are parameterized by $a \in \mathbf{R}$, the particle labels, which can be taken to be the initial data,

$$\begin{aligned} x &:= \xi_1(t,a) \quad \text{with} \quad \xi_1(0,a) = a, \\ z &:= \xi_2(t,a) \quad \text{with} \quad \xi_2(0,a) = \zeta(a,0), \end{aligned}$$

with the constraint: $\xi_2(t,a) = \zeta(\xi_1(t,a),t)$ for all (t,a).

Now, acknowledge the fact that (A 4) is a coupled system

$$\left.\begin{aligned} \frac{\mathrm{d}\xi_1}{\mathrm{d}t} &= u \quad \text{with} \quad \xi_1(0,a) = a, \\ \frac{\mathrm{d}u}{\mathrm{d}t} &= -\frac{\eta_x}{1+\eta_x^2}(\eta_{xx}\,u^2 + 2\eta_{xt}\,u + \eta_{tt}) \quad \text{with} \quad u(0,a) = u_0(a). \end{aligned}\right\} \tag{A 11}$$

The key – apparently free – parameter here is $u_0(a)$. However, in order to model the water wave with free surface ζ, this initial velocity has to satisfy the kinematic condition at the free surface: it must be determined from the Eulerian horizontal velocity. In other words, in addition to the free surface ζ, it is necessary to know the exact values of the horizontal velocity field at every point on the surface, in order to specify the appropriate initial condition for the particle velocity.

Denote the Eulerian velocity field in the fluid by $U(x, z, t)$. Then, the initial condition is determined from,

$$u_0(a) = U(a, \zeta(a, 0), 0).$$

Based on the above analysis, it is reasonable to conjecture that the solutions of the particle path equations (2.17)–(2.18) or (A 4) will be well behaved if the correct horizontal velocity at the free surface is used for the initial condition. On the other hand, Riccati equations are sensitive to changes in initial conditions: it is not clear how large the region of admissible initial data is. Suppose solutions of (A 11) with initial data $u_0(a)$ exist for all time. Will that also be true if a small numerical error is introduced into the intial data, say $u_0(a) \mapsto u_0(a) + \varepsilon$?

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